ECON6190 Section 10

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1. Let $\{X_1 \dots X_n\}$ be a sequence of i.i.d random variables with mean μ and variance σ^2. Let $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$, and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$.

   (a) Suppose $\mathbb{E}X_i^2 < \infty$, $i = 1, \dots n$. Show $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$. and $\hat{\mu} \xrightarrow{P} \mu$.

(a) Need to find $E[\hat{\mu}]$ and $\text{var}(\hat{\mu})$

$$E[\hat{\mu}] = E\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \overset{\text{linearity E[·]}}{=} \frac{1}{n}\sum_{i=1}^n \underbrace{E[X_i]}_{\text{iid, } = \mu} = \mu$$

$$\text{var}(\hat{\mu}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \overset{\text{iid}}{=} \frac{1}{n^2}\sum_{i=1}^n \underbrace{\text{var}(X_i)}_{\sigma^2} = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

By chebyshev inequality / markov (r=2):

$$P(|\hat{\mu} - \underbrace{E[\hat{\mu}]}_{\mu}| > \delta) \le \frac{E[(\hat{\mu}-\mu)^2]}{\delta^2} \overset{\text{MSE}(\hat{\mu})}{=} \frac{\text{var}(\hat{\mu})}{\delta^2} = \frac{\sigma^2}{n\delta^2} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow \hat{\mu} \xrightarrow{P} \mu$.

WTS: $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$

Notice that:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2)$$

$$= \frac{1}{n}\sum_{i=1}^n x_i^2 - 2\hat{\mu}\underbrace{\frac{1}{n}\sum_{i=1}^n x_i}_{\hat{\mu}} + \underbrace{\frac{1}{n}\sum_{i=1}^n \hat{\mu}^2}_{\hat{\mu}^2 \text{ (not indexed)}}$$

$$-2\hat{\mu}^2$$

$$= \frac{1}{n}\sum_{i=1}^n x_i^2 - \hat{\mu}^2$$

→ sample version of $\text{var}(X) = E[X^2] - (E[X])^2$

Since $E[x_i^2] < \infty$, by Khinchine's WLLN, $\frac{1}{n}\sum_{i=1}^n x_i^2 \xrightarrow{P} E[x^2]$.

We also know $\hat{\mu} \xrightarrow{P} \mu$, then by CMT,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n x_i^2 - \hat{\mu}^2 \xrightarrow{P} E[x^2] - (E[x])^2 = \text{var}(x) = \sigma^2$$

$\Rightarrow \hat{\sigma}^2 \xrightarrow{P} \sigma^2$

(b) Imposing additional assumptions if necessary, find the asymptotic distribution of

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$$

by using delta method. Carefully state your results.

Recall def: $\sigma^2 = E[X^2] - (E[X])^2$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$ ($\frac{1}{n}\sum_{i=1}^n X_i = \hat{\mu}$)

Consider a function h s.t. $h(a,b) = a - b^2$.

$\Rightarrow \sigma^2 = h(E[X^2], E[X])$, $\hat{\sigma}^2 = h\left(\frac{1}{n}\sum_{i=1}^n X_i^2, \frac{1}{n}\sum_{i=1}^n X_i\right)$

By first order Taylor expansion,

$$h\left(\frac{1}{n}\sum_{i=1}^n X_i^2, \frac{1}{n}\sum_{i=1}^n X_i\right) = h\left(E[X^2], E[X]\right) + \begin{pmatrix} \frac{\partial}{\partial a}h(a,b)\Big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \\ \frac{\partial}{\partial b}h(a,b)\Big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \end{pmatrix}^T \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n (X_i^2 - E[X^2]) \\ \frac{1}{n}\sum_{i=1}^n (X_i - E[X]) \end{pmatrix}$$

where $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}$ is in between $\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_i^2 \\ \frac{1}{n}\sum_{i=1}^n X_i \end{pmatrix}$ and $\begin{pmatrix} E[X^2] \\ E[X] \end{pmatrix}$.

Now assume $E\left[\left\|\begin{pmatrix} X^2 \\ X \end{pmatrix}\right\|^2\right] < \infty$, which requires $E[X^4] < \infty$, by multivariate CLT

$$\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n (X_i^2 - E[X^2]) \\ \frac{1}{n}\sum_{i=1}^n (X_i - E[X]) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{var}\begin{pmatrix} X^2 \\ X \end{pmatrix}\right).$$

Since we assume $E[X^4] < \infty$, $\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_i^2 \\ \frac{1}{n}\sum_{i=1}^n X_i \end{pmatrix} \xrightarrow{P} \begin{pmatrix} E[X^2] \\ E[X] \end{pmatrix}$, by WLLN.

and $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}$ is between $\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_i^2 \\ \frac{1}{n}\sum_{i=1}^n X_i \end{pmatrix}$ and $\begin{pmatrix} E[X^2] \\ E[X] \end{pmatrix}$, $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \xrightarrow{P} \begin{pmatrix} E[X^2] \\ E[X] \end{pmatrix}$.

Since $\begin{pmatrix} \frac{\partial}{\partial a}h(a,b) \\ \frac{\partial}{\partial b}h(a,b) \end{pmatrix} = \begin{pmatrix} 1 \\ -2b \end{pmatrix}$ is continuous, by CMT,

$$\begin{pmatrix} \frac{\partial}{\partial a}h(a,b)\big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \\ \frac{\partial}{\partial b}h(a,b)\big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \frac{\partial}{\partial a}h(a,b)\big|_{\binom{a}{b}=\binom{E[X^2]}{E[X]}} \\ \frac{\partial}{\partial b}h(a,b)\big|_{\binom{a}{b}=\binom{E[X^2]}{E[X]}} \end{pmatrix} = \begin{pmatrix} 1 \\ -2E[X] \end{pmatrix}.$$

$$\sqrt{n}\left(\hat{\sigma}^2 - \sigma^2\right) = \sqrt{n}\left(h\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n X_i^2, \tfrac{1}{n}\sum_{i=1}^n X_i\right) - h\left(E[X^2] - E[X]\right)\right)$$

$$= \underbrace{\begin{pmatrix} \frac{\partial}{\partial a}h(a,b)\big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \\ \frac{\partial}{\partial b}h(a,b)\big|_{\binom{a}{b}=\binom{\tilde{a}}{\tilde{b}}} \end{pmatrix}}_{\xrightarrow{P}\begin{pmatrix}1\\-2E[X]\end{pmatrix}} \underbrace{\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n (X_i^2 - E[X^2]) \\ \frac{1}{n}\sum_{i=1}^n (X_i - E[X]) \end{pmatrix}}_{\xrightarrow{d}\mathcal{N}\left(\binom{0}{0}, \operatorname{var}\binom{X^2}{X}\right)}$$

$$\xrightarrow{d} \mathcal{N}\left(\underbrace{\begin{pmatrix}1\\-2E[X]\end{pmatrix}^T}_{1\times 2}\underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{2\times 1}, \underbrace{\begin{pmatrix}1\\-2E[X]\end{pmatrix}^T}_{1\times 2} \underbrace{\operatorname{var}\begin{pmatrix}X^2\\X\end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix}1\\-2E[X]\end{pmatrix}}_{2\times 1} \right)$$

where $$\operatorname{var}\begin{pmatrix}X^2\\X\end{pmatrix} = \begin{pmatrix} \operatorname{var}(X^2) & \operatorname{cov}(X^2, X) \\ \operatorname{cov}(X^2, X) & \operatorname{var}(X) \end{pmatrix}.$$

Can simplify (not required) to $\sqrt{n}\left(\hat{\sigma}^2 - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \operatorname{var}\left((X - E[X])^2\right)\right)$

Let's find $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ w/o using delta method.

$$\hat{\sigma}^2 - \sigma^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2 - \sigma^2$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \mu + \mu - \hat{\mu})^2 - \sigma^2$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 + 2(\mu - \hat{\mu})\underbrace{\frac{1}{n}\sum_{i=1}^n (X_i - \mu)}_{=\hat{\mu} - \mu} + (\mu - \hat{\mu})^2 - \sigma^2$$

$$\underbrace{\qquad\qquad}_{-2(\mu - \hat{\mu})^2}$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 - \sigma^2 - (\mu - \hat{\mu})^2 \qquad \cdots\cdots (1)$$

$$\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \overset{①}{=} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2-\sigma^2\right) - \underbrace{\sqrt{n}(\mu-\hat{\mu})^2}_{=\sqrt{n}(\hat{\mu}-\mu)(\hat{\mu}-\mu)\,=\,o_p(1)}$$

$\sqrt{n}(\hat{\mu}-\mu)$: $O_p(1)$ b/c by CLT, $\xrightarrow{d} \mathcal{N}(0,\sigma^2)$

$(\hat{\mu}-\mu)$: $o_p(1)$ b/c $\hat{\mu}\xrightarrow{p}\mu$

$= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\underbrace{(x_i-\mu)^2}_{\text{view this as my new RV } \tilde{x}_i}-\sigma^2\right) + o_p(1)$

$\tilde{x}_i$: sample average of "$\tilde{x}_i$" $\Rightarrow$ apply CLT

$E[\tilde{x}_i] = E[(x_i-\mu)^2] = \sigma^2$

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i - E[\tilde{x}_i]$

$\Rightarrow$ by CLT, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2-\sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0,\ \text{var}\left((x-\mu)^2\right)\right)$

Q: What's the stochastic order of $\hat{\mu}-\mu$ and $\hat{\sigma}^2-\sigma^2$?

$\hat{\mu}\xrightarrow{p}\mu \Rightarrow \hat{\mu}-\mu = o_p(1)$

What's the rate of $\hat{\mu}-\mu$ converges to 0?

$$\hat{\mu}-\mu = O_p\left(\sqrt{MSE(\hat{\mu})}\right) = O_p\left(\sqrt{\frac{\sigma^2}{n}}\right) = O_p\left(\frac{1}{\sqrt{n}}\right).$$

Alternatively, by CLT, $\sqrt{n}(\hat{\mu}-\mu) \xrightarrow{d} \mathcal{N}(0,\sigma^2) \Rightarrow \sqrt{n}(\hat{\mu}-\mu) = O_p(1)$

$\Rightarrow \hat{\mu}-\mu = O_p\left(\frac{1}{\sqrt{n}}\right)$

Since $\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0,\ \text{var}\left((x-E[x])^2\right)\right) \Rightarrow \sqrt{n}(\hat{\sigma}^2-\sigma^2) = O_p(1)$

$\Rightarrow \hat{\sigma}^2-\sigma^2 = O_p\left(\frac{1}{\sqrt{n}}\right)$

$\left|\frac{\sqrt{n}(\hat{\sigma}^2-\sigma^2)}{1}\right|$ bounded in prob asymptotically